METHOD FOR THE STEADY SOLUTION OF A NONHYPERBOLIC SYSTEM OF EQUATION FOR THE MOTION OF A TWO-PHASE FLOW

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An approximate method for solving an incorrect Cauchy problem for a nonhyperbolic system of equations of a vapor-liquid flow barotropic model has been developed. The method is based on the approximation of a fluctuating parameter – the vapor phase density – by a random delta-correlated process and subsequent averaging of the stochastic equations obtained by its realization. Such a procedure allows one to reduce the matrix of the nonhyperbolic system coefficients to Hermitian form and to obtain a characteristic polynomial all the roots of which are real, i.e., it enable one to eliminate the incorrectness of the problem. The results of numerical realization have shown the steady-state character of the solution, which is in good agreement with experimental data.

1. It is well known that a basis system of equations for a two-liquid model of a vapor-liquid flow is a hyperbolic one [1]. Various simplifying physical assumptions lead to models with a "pathology," attributed to the loss of the hyperbolic character of the basis system and to instability of its solution. This problem has been considered in a number of works [1, 7-10], in which the methods for compensating nonhyperbolicity leading to a significant reduction of its domain were discussed. However, as a rule, one does not succeed in excluding it completely. Different algorithmic procedures suppressing the development of a solution instability can lead to an unestimated numerical diffusion. The source of difficulties lies in the existence of a hypothesis about the equality of phase pressures, and the usage of models with unequal pressures [11] seems quite natural. For engineering applications, this model, however, is not used in practice, since reliable information concerning the construction of closing relations when determining coefficients and right-hand sides of the system of basic equations of this model is not available.

Let us show the possibility of statistical regularization of the solution of a nonhyperbolic system of equations of the model of unequal velocities, temperatures, and equal phase pressures. For this purpose, we introduce the procedure of stochastic approximation of a fluctuating parameter and subsequent averaging of the system equations. The term "statistical regularization" was, evidently, suggested for the first time in [12, 13], in the given case, regularization with respect to the stochastic parameter of a vapor–liquid flow is implied.

2. Generally accepted operators for equation averaging with respect to the number of realizations, space, and time [1] are trivial in the sense that the statistical properties of the parameters used in these operators yield only the definition of the mean values [2]. Consider now the approach employing nontrivial statistical properties of the parameters of two-phase media. Taking into consideration space-time parameter fluctuations in two-phase flows, it is possible to apply the procedure of statistical averaging, which allows for the stochastic character of fluctuating parameters. It is well known (for example, [3]) that in many physical problems the process of parameter variation in time can be considered in the approximation of delta-correlated random processes. In particular, as applied to vapor-liquid flows, such approximation of fluctuating parameters has a fairly clear physical nature: spontaneous processes of vapor bubble generation, their collapse, formation of films, shells can be treated as jumps of the statistical mean parameter values for the delta-correlated process under consideration. For the time distribution parameter values of the flow, resulting from the effect of the simultaneous action of a combination of factors, the Gaussian character of fluctuations can be considered acceptable. Such an approximation is widely used, for example, for the consideration of turbulent flows [4-6].

Assume the system of equations [9] as a basis one:

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$$\frac{D_k W_k}{Dt} + \frac{1}{\rho_k}\frac{\partial p_k}{\partial z} = \Pi_1;$$

$$\frac{D_k h_k}{Dt} + a_k^2 \frac{\partial W_k}{\partial z} + \frac{a_k^2}{\varphi_k}\frac{D_k \varphi_k}{Dt} = \Pi_2; \tag{1}$$

$$\frac{D_k p_k}{Dt} + a_k^2 \rho_k^2 \frac{\partial W_k}{\partial z} + \frac{a_k^2 \rho_k}{\varphi_k}\frac{D_k \varphi_k}{Dt} = \Pi_3; \tag{2}$$

$$a_k^{-2} = \frac{\partial \rho_k}{\partial p_k} + \frac{1}{\rho_k}\frac{\partial \rho_k}{\partial h_k}, \tag{3}$$

where $D_k/Dt = \partial/\partial t + W_k(\partial/\partial z)$, p_k, h_k, φ_k, are pressure, enthalpy, volume concentration of phases; a_k is the propagation velocity of acoustic disturbances over the phase; Π_1, Π_2, Π_3, are the right-hand sides of equations, $k = g, l$ (g is the steam phase, l is the liquid one). Assuming $p_g = p_l = p$, after transformations, we obtain the following system of equations:

$$\frac{\partial W_k}{\partial t} + W_k \frac{\partial W_k}{\partial z} + \frac{1}{\rho_k}\frac{\partial p}{\partial z} = \Pi_4, \tag{4}$$

$$\frac{\partial p}{\partial t} + R\sum_k \varphi_k \frac{\partial W_k}{\partial z} + R\sum_k \frac{\varphi_k W_k}{\rho_k a_k^2}\frac{\partial p}{\partial z} + R(W_g - W_l)\frac{\partial \varphi}{\partial z} = \Pi_5; \tag{5}$$

$$\frac{\partial \varphi}{\partial t} + \varphi\left(1 - \frac{\varphi R}{\rho_g a_g^2}\right)\frac{\partial W_g}{\partial z} - \frac{R\varphi(1-\varphi)}{\rho_g a_g^2}\frac{\partial W_l}{\partial z} +$$

$$+ \frac{\varphi}{\rho_g a_g^2}\left(W_g - R\sum_k \frac{\varphi_k W_k}{\rho_k a_k^2}\right)\frac{\partial p}{\partial z} + \left[W_g - \frac{\varphi R(W_g - W_l)}{\rho_g a_g^2}\right]\frac{\partial \varphi}{\partial z} = \Pi_6; \tag{6}$$

$$\frac{\partial h_k}{\partial t} + \frac{R}{\rho_k}\sum_k \varphi_k \frac{\partial W_k}{\partial z} - \frac{1}{\rho_k}\left(W_k - R\sum_k \frac{\varphi_k W_k}{\rho_k a_k^2}\right)\frac{\partial p}{\partial z} + \frac{R(W_g - W_l)}{\rho_k}\frac{\partial \varphi}{\partial z} + W_k \frac{\partial h_k}{\partial z} = \Pi_7, \tag{7}$$

where

$$R = \left(\sum_k \frac{\varphi_k}{\rho_k a_k^2}\right)^{-1}, \quad \varphi = \varphi_g, \quad \varphi_l = 1 - \varphi.$$

The vector of unknowns of the system (4)-(7): $S = S(W_g, W_l, p, \varphi, h_g, h_l)$; the following determinant corresponds to the matrix of this system:

$$\begin{vmatrix} (m_{11} - \nu) & 0 & m_{13} & 0 & 0 & 0 \\ 0 & (m_{22} - \nu) & m_{23} & 0 & 0 & 0 \\ m_{31} & m_{32} & (m_{33} - \nu) & m_{34} & 0 & 0 \\ m_{41} & m_{42} & m_{43} & (m_{44} - \nu) & 0 & 0 \\ m_{51} & m_{52} & m_{53} & m_{54} & (m_{55} - \nu) & 0 \\ m_{61} & m_{62} & m_{63} & m_{64} & 0 & (m_{66} - \nu) \end{vmatrix} \tag{8}$$

where ν stands for the characteristic directions of the system, $m_{11} = W_g$,

$$m_{13} = \rho_g^{-1}, \quad m_{22} = W_l, \quad m_{23} = \rho_l^{-1}, \quad m_{31} = R\varphi, \quad m_{32} = R(1-\varphi),$$

$$m_{33} = R\sum_k \frac{\varphi_k W_k}{\rho_k a_k^2}, \quad m_{34} = R(W_g - W_l),$$

$$m_{41} = \varphi\left(1 - \frac{R\varphi}{\rho_g a_g^2}\right), \quad m_{42} = -\varphi(1-\varphi)\frac{R}{\rho_g a_g^2},$$

$$m_{43} = \frac{\varphi}{\rho_g a_g^2}\left(W_g - R\sum_k \frac{\varphi_k W_k}{\rho_g a_g^2}\right), \quad m_{44} = W_g - \frac{\varphi R}{\rho_g a_g^2}(W_g - W_l),$$

$$m_{51} = \frac{\varphi R}{\rho_g}, \quad m_{52} = \frac{R(1-\varphi)}{\rho_g}, \quad m_{53} = -\frac{1}{\rho_g}\left(W_g - R\sum_k \frac{\varphi_k W_k}{\rho_k a_k^2}\right),$$

$$m_{54} = \frac{R}{\rho_g}(W_g - W_l), \quad m_{55} = W_g, \quad m_{61} = \frac{\varphi R}{\rho_l}, \quad m_{62} = \frac{R(1-\varphi)}{\rho_l},$$

$$m_{63} = -\frac{1}{\rho_l}\left(W_l - R\sum_k \frac{\varphi_k W_k}{\rho_k a_k^2}\right), \quad m_{64} = \frac{R(W_g - W_l)}{\rho_l}.$$

The appropriate characteristic polynomial has the form

$$(W_g - \nu)(W_l - \nu)\left\{(W_g - \nu)^2 (W_l - \nu)^2 - \frac{\varphi\rho_l (W_l - \nu)^2}{\frac{\varphi\rho_l}{a_g^2} + \frac{(1-\varphi)\rho_g}{a_l^2}} - \frac{(1-\varphi)\rho_g (W_g - \nu)^2}{\frac{\varphi\rho_l}{a_g^2} + \frac{(1-\varphi)\rho_g}{a_l^2}}\right\} = 0, \tag{9}$$

from which it follows that $\nu_1 = W_g$, $\nu_2 = W_l$ are its two roots, whereas the rest are the roots of the polynomial enclosed in curly brackets expression (9). The latter ones include complex roots, and thus the Cauchy problem becomes incorrect. The suggested approach to the hyperbolization of system (4)-(7) consists in reducing the matrix of its coefficients to Hermitian form, characteristic directions of which are real. This result is obtained by the stochastic approximation of a certain flow-parameter and the subsequent averaging of equations with respect to its realizations. Owing to the statistical character of processes in two-phase flows, fluctuations of density of the vapor phase always occur, and to a lesser degree, those of the liquid one. Let us present the density of the steam phase as a random function equal to the sum of the averaged and pulsating terms:

$$\rho_g = \langle \rho_g \rangle + \delta\rho_g(t), \tag{10}$$

it being $\langle \delta\rho_g \rangle = 0$. Since $\delta\rho_g << \langle \rho_g \rangle$, Eq. (10) can be presented in the following form:

$$\rho_g = \langle \rho_g \rangle \left(1 + \frac{\delta\rho_g}{\langle \rho_g \rangle}\right) \approx \langle \rho_g \rangle \exp\left(\frac{\delta\rho_g}{\langle \rho_g \rangle}\right). \tag{11}$$

By analogy, using the known expansion, we present the value of ρ^{-1}_g:

$$\frac{1}{\rho_g} \approx \frac{1}{\langle \rho_g \rangle}\left(1 - \frac{\delta\rho_g}{\langle \rho_g \rangle}\right) \approx \frac{1}{\langle \rho_g \rangle}\exp\left(-\frac{\delta\rho_g}{\langle \rho_g \rangle}\right). \tag{12}$$

Since the vapor phase fluctuations are of a pulsating (wave) character [14] and can be described by the delta-correlated Gaussian process, we introduce the complex correlation function

$$\langle \delta\rho_g(t_1)\,\delta\rho_g(t_2) \rangle = 2i\sigma^2\delta(t_1 - t_2), \tag{13}$$

where σ is the dispersion of the vapor phase density. In the given case, ρ_g has been chosen as a stochastic parameter due to the fact that in the case when strong disturbances of other parameters of the flow are absent, the density of the vapor phase and pressure have the highest degree of correlation. In this case, correlation between the density and concentration fields can be neglected. In the approximation of mutual statistical independence of the parameters W_k, φ_k, h_k, ρ_l, φ_l, and p, assuming $a^2_k \approx$ const, we should further perform averaging of the system of equations with respect to the realizations of the random process ρ_g. Prior to that, let us transform determinant (8) and the appropriate equations of the system (4)-(7). Since expansion of the determinant into the m_{55} and m_{66} elements yields real roots, it is sufficient to consider the determinant $|m_{ij}|$ (i, j = 1, 2, 3, 4), producing complex characteristic directions. Let us take the common factors R, $\varphi R(\rho_g a^2_g)^{-1}$, $a^2_g R^{-1}$, ρ^{-1}_l outside the sign of this determinant, which corresponds to division of the appropriate system equations by these values. As a result of these transformations, we obtain the determinant

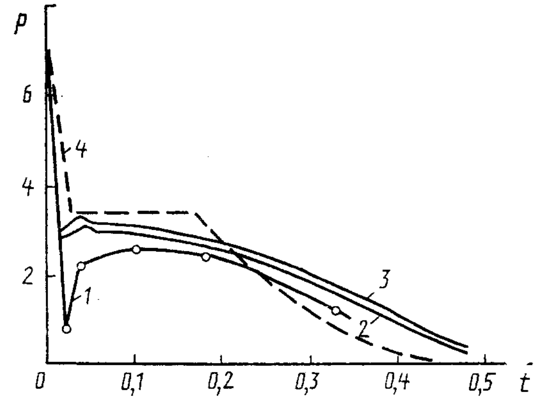

Fig. 1. Experimental and predicted pressure values: 1) experiment; 2) the result [9]; 3) the result of the present work; 4) homogeneous equilibrium model; p, MPa; t, sec.

$$\begin{vmatrix} \dfrac{W_g - \nu}{W_l - \nu} & 0 & \dfrac{1}{\rho_g} \\ 0 & \dfrac{(W_l - \nu)\,\varphi\rho_l}{(1-\varphi)(W_g - \nu)} & 1 \\ \rho_g & 1 & \dfrac{W_g - \nu}{\varphi}\left(\sum_k \dfrac{\varphi_k W_k}{\rho_k a_k^2} - \dfrac{\nu}{R}\right) - \dfrac{(W_g - \nu)(W_g - W_l)}{\rho_g a_g^2} \end{vmatrix} \tag{14}$$

which gives the polynomial in the curly brackets of expression (9). As a result of the transformations of the system equations in compliance with the form of determinant (14) and their averaging with respect to the realization of the process ρ_g, terms appear which contain stochastic nonlinearities: $\langle\rho^{-1}_g(\partial p/\partial z)\rangle$, $\langle\rho_g(\partial W_g/\partial z)\rangle$. We reveal them, using the Furuttsu–Novikov equation and correlation (13). Taking into account (11) and (12), we obtain, respectively

$$\left\langle \rho_g \frac{\partial W_g}{\partial z}\right\rangle \approx \langle \rho_g\rangle \left\langle \exp\left(\frac{\delta\rho_g}{\langle\rho_g\rangle}\right)\frac{\partial W_g}{\partial z}\right\rangle \approx$$

$$\approx \langle\rho_g\rangle\left\{\left(1 + \frac{i\sigma^2}{\langle\rho_g\rangle^2}\right)\frac{\partial\langle W_g\rangle}{\partial z} - \frac{2\sigma^2}{\langle\rho_g\rangle}\cdot\frac{\partial^2\langle W_g\rangle}{\partial z^2}\right\},$$

$$\left\langle \frac{1}{\rho_g}\frac{\partial p}{\partial z}\right\rangle \approx \frac{1}{\langle\rho_g\rangle}\left\langle \exp\left(-\frac{\delta\rho_g}{\langle\rho_g\rangle}\right)\frac{\partial p}{\partial z}\right\rangle \approx$$

$$\approx \frac{1}{\langle\rho_g\rangle}\left\{\left(1 - \frac{i\sigma^2}{\langle\rho_g\rangle^2}\right)\frac{\partial\langle p\rangle}{\partial z} - \frac{2\sigma^2}{\langle\rho_g\rangle}\frac{\partial^2\langle p\rangle}{\partial z^2}\right\}.$$

Multiplying the averaged equation (4) by $\langle\rho_g\rangle^2$, we finally get the matrix of coefficients of the averaged system in Hermitian form and the determinant

$$\begin{vmatrix} \dfrac{\langle W_g\rangle - \nu}{W_l - \nu} & 0 & \langle\rho_g\rangle\left(1 - \dfrac{i\sigma^2}{\langle\rho_g\rangle^2}\right) \\ 0 & \dfrac{(W_l - \nu)\varphi\rho_l}{(1-\varphi)(\langle W_g\rangle - \nu)} & 1 \\ \langle\rho_g\rangle\left(1 + \dfrac{i\sigma^2}{\langle\rho_g\rangle^2}\right) & 1 & \dfrac{\langle W_g\rangle - \nu}{\varphi}\left(\left\langle\sum_k \dfrac{\varphi_k W_k}{\rho_k a_k^2} - \dfrac{\nu}{R}\right\rangle\right) - \dfrac{(\langle W_g\rangle - \nu)(\langle W_g\rangle - W_l)}{\langle\rho_g\rangle a_g^2} \end{vmatrix}$$

which possesses Hermitian character with accuracy up to the operator $(-2\sigma^2 <\rho_g> ^{-1}\partial/\partial z)$, which creates the second derivatives.

3. It is known [14] that p(z), $W_g(z)$ are the functions close to linear ones. Allowing also for the fact that for the second derivatives the determinant is $\sigma^2 << \langle\rho_g\rangle$, one can assume that it is possible to neglect terms which contain the second derivatives. But this assumption, however, is not evident and requires verification. For this purpose, the effect exerted on the numerical results by terms comprising the second derivatives of velocity and pressure at different values of the dispersion σ has been investigated. The largest relative discrepancy in the results, with account for and account for the second derivatives, does not exceed 0.20.

Figure 1 presents the result of the numerical realization of the proposed approach, as applied to the prediction of pressure in the case of boiling-up water effluxing from an unheated horizontal tube filled by subcooled (up to the saturation temperature) water at the following initial parameters: pressure p_0 = 7 MPa, temperature T_0 = 513 K [9]. The solution obtained is of a steady-state character and satisfactorily agrees with the experimental data.

NOTATION

a, velocity of propagation of acoustic disturbances over the phase, [m·sec^{-1}]; D/Dt, substantial derivative; h, enthalpy, J; i, imaginary unit; m, determinant element; Π, the right-hand side of an equation; p, pressure, Pa; S, vector of unknowns; T, temperature, K; t, time, sec; W, phase velocity, m·sec^{-1}; z, coordinate, m; δ, delta function; ν, root of a characteristic equation; ρ, phase density, kg·m^{-3}; σ, gas phase density dispersion, kg·m^{-3}; φ, volumetric phase concentration. Indices: g, gas phase; l, liquid phase; 0, initial value of a parameter.

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